Cohen-Macaulay type of endomorphism rings of abelian varieties over finite fields

...or...

when an abelian variety met Bruns-Herzog's book.

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AGC^{2}T 2023 - 6 June 2023.

- Let A be an **abelian variety** over $\mathbb{F}_q$, $q = p^a$, of dimension g.
- $\mathrm{End}_{\mathbb{F}_q}(A)$ is a free $\mathbb{Z}$-module of finite rank ...
- ... $\mathrm{End}_{\mathbb{F}_q}(A) \subset \mathrm{End}_{\mathbb{F}_q}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Denote by $\pi_A \in \mathrm{End}_{\mathbb{F}_q}(A)$ the Frobenius endomorphism of A...
- ... and by $h_A(x)$ the **characteristic polynomial** of π_A acting on

$$\pi_A \curvearrowright T_\ell A = \varprojlim A[l^n] \simeq_{\mathbb{Z}_\ell} \mathbb{Z}_\ell^{2g}, \quad \text{for a prime } \ell \neq p.$$

- Ex. $E/\mathbb{F}_5 : Y^2 = X^3 + X \rightsquigarrow h_E(x) = x^2 - 2x + 5$.
- Ex. $C/\mathbb{F}_3 : Y^2 = X^6 + X + 1 \rightsquigarrow h_{\mathrm{Jac}(C)}(x) = x^4 + 3x^3 + 6x^2 + 9x + 9$.

Abelian varieties : endomorphism algebra

- Some facts (Tate + Weil conjectures):
  - h_A does not depend on the choice of ℓ.
  - $h_A \in \mathbb{Z}[x]$ of degree $2g$.
  - $A/\mathbb{F}_q$ and $B/\mathbb{F}_q$ are $\mathbb{F}_q$-isogenous $\iff h_A = h_B$.
  - h_A is squarefree (i.e. no repeated $\mathbb{C}$-roots) $\iff \mathrm{End}_{\mathbb{F}_q}(A)$ is commutative.
- From now on:
  - We assume that h_A is **squarefree**.
  - We identify $\mathrm{End}_{\mathbb{F}_q}(A) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[x]/h_A = \mathbb{Q}[\pi]$ by $\pi_A \mapsto \pi$.
- Note:
  - $K = \mathbb{Q}[\pi]$ is a **étale $\mathbb{Q}$-algebra** (i.e. a finite product of number fields).
  - $\mathbb{Z}[\pi, q/\pi] \subseteq \mathrm{End}_{\mathbb{F}_q}(A) \subseteq \mathcal{O}_K$ are orders in K (an **order** R is a subring $R \subset K$ such that $R \simeq_{\mathbb{Z}} \mathbb{Z}^{\dim_{\mathbb{Q}} K}$).

Orders and fractional ideals in étale $\mathbb{Q}$-algebras

- Let R be an order in a étale $\mathbb{Q}$-algebra K.
- A **fractional** R-**ideal** is a sub-R-module $I \subset K$ such that $I \simeq_{\mathbb{Z}} \mathbb{Z}^{\dim_{\mathbb{Q}} K}$.
- Given fr. R-ideals I, J then

$$(I : J) = \{a \in K : aJ \subseteq I\} \quad \text{and} \quad I^t = \left\{a \in K : \mathrm{Tr}_{K/\mathbb{Q}}(aI) \subseteq \mathbb{Z}\right\}$$

  are also fr. R-ideals.
- We have $(I : I)^t = I \cdot I^t$.
- A fr. R-ideal I is invertible if $I(R : I) = R$ ...
- ... or, equivalently, $I_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$-modules for every $\mathfrak{p}$ maximal R-ideal. ($R_{\mathfrak{p}}$ is the completion of R at $\mathfrak{p}$)
- If I is invertible, then $(I : I) = R$.

Cohen-Macaulay type and Gorenstein orders

- Def: The **(Cohen-Macaulay) type** of R at a maximal ideal $\mathfrak{p}$ is

$$\mathrm{type}_{\mathfrak{p}}(R) := \dim_{R/\mathfrak{p}} \frac{R^t}{\mathfrak{p}R^t}.$$

- Def: R is **Gorenstein** at $\mathfrak{p}$ if $\mathrm{type}_{\mathfrak{p}}(R) = 1$.
- Remark: these definitions coincides with the 'usual' ones.
- Ex: monogenic $\mathbb{Z}[\alpha]$ and maximal $\mathcal{O}_K$ orders are Gorenstein. (also $\mathbb{Z}[\pi, q/\pi]$ for AVs).
- Ex: pick a prime $\ell \in \mathbb{Z}$. Then $\mathrm{type}_{\ell\mathcal{O}_K}(\mathbb{Z} + \ell\mathcal{O}_K) = \dim_{\mathbb{Q}} K - 1$.

Classification for orders of type ≤ 2

Theorem

Let $\mathfrak{p}$ be a maximal ideal of R, and I a fr. R-ideal with $(I:I) = R$.

1. *If $\operatorname{type}_{\mathfrak{p}}(R) = 1$ (Gorenstein) then $I_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$-modules.*
2. *If $\operatorname{type}_{\mathfrak{p}}(R) = 2$ then either $I_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$ or $I_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^t$ as $R_{\mathfrak{p}}$-modules.*

Part 1 is contained (in a much more general form) in the "Ubiquity" paper by H. Bass.
Part 2 is new, and we give a proof.

Lemma

Let U, V, W be vectors spaces (over some field). Assume that $\dim W \geq 2$, and let $m : U \otimes V \to W$ be a surjective map. Then:

1. *$\exists u \in U$ such that $\dim(m(u \otimes V)) \geq 2$, or*
2. *$\exists v \in V$ such that $\dim(m(U \otimes v)) \geq 2$.*

Proof of Part 2

- Put $U = I/\mathfrak{p}I$, $V = I^t/\mathfrak{p}I^t$ and $W = R^t/\mathfrak{p}R^t$.
- By assumption $R^t = I \cdot I^t$, so the map $m : U \otimes V \to W$ induced by multiplication $I \times I^t \to R^t$ is surjective.
- Moreover, $\dim W = 2$ (because of the assumption on the type).
- By the Lemma:
  1. $\exists x \in I$ such that $m((x + \mathfrak{p}I) \otimes V) = \frac{xI^t + \mathfrak{p}R^t}{\mathfrak{p}R^t}$ equals W.
     By Nakayama's lemma: $I^t_{\mathfrak{p}} \simeq R^t_{\mathfrak{p}} \iff R_{\mathfrak{p}} \simeq I_{\mathfrak{p}}, \ldots$
  2. ...or, $\exists y \in I^t$ such that $U \otimes m(U \otimes (y + \mathfrak{p})I^t) = W$ implying $I^t_{\mathfrak{p}} \simeq R_{\mathfrak{p}} \iff I_{\mathfrak{p}} \simeq R^t_{\mathfrak{p}}$.

Back to AVs: Categorical equivalence(s)

Fix a squarefree characteristic poly $h(x)$ of Frobenius π over $\mathbb{F}_q$.
Put $K = \mathbb{Q}[x]/h = \mathbb{Q}[\pi]$.
Let $\mathscr{I}_h$ be the corresponding isogeny class.

Theorem

Assume that $q = p$ is prime or that $\mathscr{I}_h$ is ordinary.
Then there is an **equivalence** *of categories*

$$\left\{ \ \mathscr{I}_h \text{ with } \mathbb{F}_q\text{–morphisms} \ \right\}$$
$$\updownarrow$$
$$\left\{ \text{fr. } \mathbb{Z}[\pi, q/\pi]\text{-ideals with linear morphisms} \right\}$$

Moreover, if $A \mapsto I$ then $A^\vee \mapsto \overline{I}^t$, where $\overline{\cdot}$ is defined by $\overline{\pi} = q/\pi$ (the CM-involution).

References: Deligne, Howe, Centeleghe-Stix, Bergström-Karemaker-M.

AVs: self-duality

Theorem (Springer-M.)

$\mathscr{I}_h$ and $K = \mathbb{Q}[\pi] = \mathbb{Q}[x]/h$ as before.
Let R be an order in K and $\mathfrak{p}$ a maximal ideal of R (possibly but not necessarily above p). Assume:

$$R = \overline{R}, \quad \mathfrak{p} = \overline{\mathfrak{p}}, \quad \textit{and} \quad \mathrm{type}_{\mathfrak{p}}(R) = 2.$$

Then for every $A \in \mathscr{I}_h$ such that $\mathrm{End}(A) = R$ we have that $A \not\simeq A^\vee$. In particular, such an A cannot be principally polarized nor a Jacobian.

Proof: Say that $A \mapsto I$. Hence $A^\vee \mapsto \overline{I}^t$.
By the Classification: either $I_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$ or $I_{\mathfrak{p}} \simeq R^t_{\mathfrak{p}}$.
In the first case: $\overline{I}^t_{\mathfrak{p}} = \overline{I}^t_{\overline{\mathfrak{p}}} \simeq R^t_{\mathfrak{p}} \not\simeq R_{\mathfrak{p}}$.
Similarly, in the second: $\overline{I}^t_{\mathfrak{p}} = \overline{I}^t_{\overline{\mathfrak{p}}} \simeq R_{\mathfrak{p}} \not\simeq R^t_{\mathfrak{p}}$.
In both cases: $I \not\simeq \overline{I}^t \iff A \not\simeq A^\vee$.

Some stats and refs

Soon on the LMFDB there will be tables of isomorphism classes of AVs/$\mathbb{F}_q$.
Over 615269 isogeny classes for $1 \leq g \leq 5$ and various q, we encountered

- 3.914.908 commutative endomorphism rings, of which:
- 72.6% satisfy $R = \overline{R}$;
- 10.3% satisfy $R = \overline{R}$ and are non-Gorenstein;
- 7.4% satisfy $R = \overline{R}$, are non-Gorenstein and the Theorem applies.

References:

- *Cohen-Macaulay type of orders, generators and ideal classes*
  `https://arxiv.org/abs/2206.03758`
- *Abelian varieties over finite fields and their groups of rational points*
  with Caleb Springer,
  `https://arxiv.org/abs/2211.15280`
- Magma package for étale $\mathbb{Q}$-algebras
  `https://github.com/stmar89/AlgEt` (also in Magma 2-28.1, without documentation...)

Thank you!